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The Chebyshev Theory of a Variation of L Approximation*

YING GUANG SHI

Computing Center, Chinese Academy of Sciences,
P.O. Box 2719, Beijing, China 100080

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A new method of approximation is proposed which maintains almost all of the essentials of the Chebyshev theory of best uniform approximation, while also using an L-type measure of approximation.

1. INTRODUCTION

In a recent paper Pinkus and Shisha [2] proposed a new method of approximation which maintains many of the essentials of the classical theory of best uniform approximation, while also using an L^q-type $(1 \leqslant q < \infty)$ measure of approximation. But, as they mention, their "distance" function is not derived from a norm. Moreover, the Chebyshev's alternation characterization is not complete for the gauge $|||\cdot|||$ [2, Theorem 3.1], and a best approximation does not necessarily exist for the gauge $|||\cdot|||^*$ [2, Theorem 2.5].

In this paper we propose another new method of approximation which is based on a norm and maintains almost all of the essentials of the Chebyshev theory of best uniform approximation, while also using an L-type measure of approximation.

Let $C[a, b]$ denote the class of real-valued functions continuous on $[a, b]$. For $f \in C[a, b]$ we define

$$\|f\| = \sup \left\{ \left| \int_c^d f(x)\, dx \right| : a \leqslant c \leqslant d \leqslant b \right\}. \tag{1}$$

It is easy to see that the supremum is attained. In the next section we shall see that this is indeed a norm.

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Let G be an n-dimensional subspace of $C[a, b]$. Our problem is, given $f \in C[a, b]$, find a $p \in G$ such that

$$\|f-p\| = \inf_{q \in G} \|f-q\|.$$

Such a function p (if any) is defined to be a best approximation to f from G.

In Section 2 we shall discuss some properties of this norm. Sections 3 and 4 are devoted to developing characterizations and uniqueness of best approximation, respectively.

2. Preliminaries

First we introduce some notation and definitions. Define $X := \{I=(c, d): I \subset [a, b]\}$. We adopt the convention that X contains the unique "zero" element $0=\phi$. If $I=(c, d) \in X \backslash \{0\}$, we write $I^- = c$ and $I^+ = d$. $I_m \to I := I_m^- \to I^-$ and $I_m^+ \to I^+$. In what follows we always assume that $f \in C[a, b]$.

For ease of notation we set

$$f(I) := \int_I f(x)\, dx$$

and

$$X_f := \{I \in X: |f(I)| = \|f\|\}.$$

With this notation (1) may be rewritten as

$$\|f\| = \sup_{I \in X} |f(I)|. \tag{2}$$

Lemma 1. *If* $I \in X_f$, *then* $I^-, I^+ \in Z(f) \cup \{a, b\}$, *where* $Z(f) = \{x \in [a, b]: f(x)=0\}$.

Proof. Suppose on the contrary that $I^- \notin Z(f) \cup \{a, b\}$. We assume without loss of generality that $f(I) = \|f\|$. If $f(I^-) > 0$ (<0), then $f((I^- - t, I^+)) > f(I) = \|f\|$ for $t > 0$ (<0) sufficiently small. This contradiction proves $I^- \in Z(f) \cup \{a, b\}$. Similarly $I^+ \in Z(f) \cup \{a, b\}$. ∎

Theorem 1. $\|\cdot\|$ *is a norm and* $\|f\| = \sup\{|f(I)|: I^-, I^+ \in Z(f) \cup \{a, b\}\}$.

Proof. It is easy to check that $\|\cdot\|$ is a norm. For example, for the triangle inequality we see that

$$\|f+g\| = \sup_{I\in X} |f(I)+g(I)| \leqslant \sup_{I\in X} |f(I)| + \sup_{I\in X} |g(I)| = \|f\| + \|g\|.$$

The latter conclusion of the theorem follows directly from Lemma 1. ∎

By Theorem 1 the existence theorem in [1, p. 20] guarantees that to each $f\in C[a, b]$ there exists at least one function $p\in G$ which best approximates f.

Now we give some properties of X_f.

LEMMA 2. *Let $I, J\in X_f$.*

(a) *If $f(I)=f(J)$ with $I\cap J\neq 0$, then $f(I\backslash J)=f(J\backslash I)=0$ and $f(I\cap J)=f(I\cup J)=f(I)$;*

(b) *If $f(I)=f(J)$ with $I\supset J$, then $f((I^-, J^-))=f((J^+, I^+))=0$;*

(c) *If $f(I)=-f(J)$ with $I^-\leqslant J^-$ and $I^+\leqslant J^+$, then $f(I\cap J)=0$ and $f(I\backslash J)=-f(J\backslash I)=f(I)$;*

(d) *If $f(I)=-f(J)$ with $I\supset J$, then $f((I^-, J^-))=f((J^+, I^+))=f(I)$.*

Proof. We assume without loss of generality that $f(I)=\|f\|$. Denote $L=(I^-, J^-)$ and $R=(J^+, I^+)$.

(a) Since $f(I\backslash J)=f(I)-f(I\cap J)=\|f\|-f(I\cap J)\geqslant 0$ and $f(I\backslash J)=f(I\cup J)-f(J) = f(I\cup J)-\|f\| \leqslant 0$, $f(I\backslash J) = 0$. Similarly $f(J\backslash I) = 0$. Whence $f(I\cap J)=f(I\cup J)=f(I)$.

(b) It follows from (a) that $f(L)+f(R)=0$. Since $f(L)=f(L\cup J)-f(J)\leqslant 0$ and $f(R)=f(J\cup R)-f(J)\leqslant 0$, $f(L)=f(R)=0$.

(c) Since $f(I\cap J)=f(I)-f(I\backslash J)\geqslant 0$ and $f(I\cap J)=f(J)-f(J\backslash I)\leqslant 0$, $f(I\cap J)=0$. Hence $f(I\backslash J)=-f(J\backslash I)=f(I)$.

(d) That $f(L)+f(R)=f(I)-f(J)=2f(I)$ implies $f(L)=f(R)=f(I)$. ∎

LEMMA 3. *Let I, J, and K satisfy $I^+=K^-$ and $K^+=J^-$. Let $I, J\in X_f$. Then*

(a) *If $f(I)=f(J)$, then $f(K)=-f(I)$;*

(b) *If $f(I)=-f(J)$, then $f(K)=0$.*

Proof. As before, we assume $f(I)=\|f\|$.

(a) Since $f(K)=f(I\cup K\cup J)-f(I)-f(J)\leqslant -f(I)$, $f(K)=-f(I)$.

(b) Since $f(K)=f(I\cup K)-f(I)\leqslant 0$ and $f(K)=f(J\cup K)-f(J)\geqslant 0$, $f(K)=0$. ▮

In order to describe the further properties of X_f we need the following definitions.

Definition 1. Let $f\neq 0$. An $I\in X_f$ is said to be a definite interval of f if there is no $J\subset I$ satisfying $f(J)=-f(I)$. The set of all definite intervals of f is denoted by X_f^*.

An $I\in X_f^*$ is said to be a maximal (resp. minimal) definite interval of f if there is no $J\supset I$ (resp. $J\subset I$) satisfying $J\in X_f^*$ and $J\neq I$. The set of all maximal (resp. minimal) definite intervals of f is denoted by X_f^M (resp. X_f^m).

Remark. By the definition and Lemma 2 it is easy to see that if I, $J\in X_f^*$ with $f(I)=f(J)$ and $I\cap J\neq 0$ then $I\cup J\in X_f^*$.

Definition 2. $\{I_1, ..., I_m\}\subset X\backslash\{0\}$ is said to be weakly increasing if

(a) $I_i^- < I_{i+1}^-$ and $I_i^+ < I_{i+1}^+$, $i=1, ..., m-1$;

(b) $I_i^+ < I_{i+2}^-$, $i=1, ..., m-2$.

If I and J are nonempty subintervals of $[a, b]$, $I<J$ means that $x<y$ for all $x\in I$ and all $y\in J$.

$\{I_1, ..., I_m\}\subset X\backslash\{0\}$ is said to be increasing if $I_1<\cdots<I_m$.

A system of extended intervals $I_1, ..., I_m$, i.e., $I_i\in X$ or $I_i=[x, x]:=x$, $x\in[a, b]$, is said to be increasing if $I_1<\cdots<I_m$.

Remark. It is easy to see that if $\{I_1, ..., I_m\}$ is increasing (resp. weakly increasing) then any subset $\{I_{i_k}\}$ of $\{I_1, ..., I_m\}$ with $i_1<i_2<\cdots$ is also increasing (resp. weakly increasing).

Lemma 4. *Let $f\neq 0$. Each $I\in X_f$ contains an interval $J\in X_f^*$ with $f(I)=f(J)$.*

Proof. Suppose to the contrary that for some $I\in X_f$ such an interval J does not exist. Then for $I_0\equiv I$ there exists a $J_1\subset I_0$ satisfying $f(J_1)=-f(I)$. By Lemma 2 we have that $I_0^- < J_1^- < J_1^+ < I_0^+$ and $f(I_1)=f(I)$, where $I_1=(I_0^-, J_1^-)$ satisfies $J_1\subset I_0\backslash I_1$. We can by induction obtain $\{I_i\}$ and $\{J_i\}$ which satisfy $I_i\subset I_{i-1}$, $J_i\subset I_{i-1}\backslash I_i$, $f(I_i)=f(I)$, and $f((J_i)=-f(I)$, $i=1, 2, ...$ It is easy to see that the $\{J_i\}$ are all disjoint, a contradiction. This completes the proof of the lemma. ▮

Lemma 5. *Each $I\in X_f^*$ must be contained in a unique interval $J\in X_f^M$.*

Proof. Put

$$J^- = \inf\{K^- : K^+ = I^+ \text{ and } K \in X_f^*\},$$
$$J^+ = \sup\{K^+ : K^- = I^- \text{ and } K \in X_f^*\}.$$

Denote $L = (J^-, I^+)$, $R = (I^-, J^+)$, and $J = L \cup R$.

First, we see that $f(L) = f(R) = f(I)$, whence $f(J) = f(I)$. Thus $J \in X_f$.

Next, we prove that $J \in X_f^*$. Suppose to the contrary that there is a $K \subset L$ satisfying $f(K) = -f(I)$. Thus, if $K^- = L^-$, then $f((K) = f(L \cap K) = 0$ by Lemma 2, and if $K^- > L^-$, then there is a $K_1 \in X_f^*$ such that $K_1 \supset (K \cup L)$ and $K_1^+ = L^+$. Both of them are impossible. This contradiction proves $L \in X_f^*$. Similarly $R \in X_f^*$. Then $J \in X_f^*$.

On the other hand, suppose that there is a $K \in X_f^*$ with $K \supset J$. Then it is easy to check that $K_1 := (K^-, L^+) \supset L$ and $K_1 \in X_f^*$. So we must have $K_1 = L$. Similarly $(L^-, K^+) = R$. Thus $K = J$ and $J \in X_f^M$.

The uniqueness is obvious. ∎

LEMMA 6. *Let $I, J \in X_f^M$ satisfy $f(I) = f(J)$ with $I \neq J$ and $I^- \leqslant J^-$. Then*

(a) $I \cap J = 0$;

(b) *There is a $K \in X_f^M$ satisfying $f(K) = -f(I)$ and for which $\{I, K, J\}$ is weakly increasing.*

Proof. (a) If $I \cap J \neq 0$, by the remark after Definition 1 we have $I \cup J \in X_f^*$, which is impossible because $J \in X_f^M$. So $I \cap J = 0$.

(b) By Lemma 3 we see that $f(K_1) = -f(I)$, where $K_1 := (I^+, J^-)$. Using Lemma 4 we may choose a $K_2 \in X_f^*$ with $K_2 \subset K_1$ and $f(K_2) = -f(I)$. By virtue of Lemma 5 we can find a $K \in X_f^M$ with $K \supset K_2$ and $f(K) = -f(I)$. Clearly $\{I, K, J\}$ is weakly increasing. ∎

THEOREM 2. *X_f^M is finite. Moreover $X_f^M = \{I_i\}_1^N$ with $I_1^- \leqslant \cdots \leqslant I_N^-$ is weakly increasing and satisfies $f(I_{i+1}) = -f(I_i)$, $i = 1, 2, ..., N-1$.*

Proof. By Lemma 6 the intervals in $\{J \in X_f^M : f(J) > 0\}$ and the intervals in $\{K \in X_f^M : f(K) < 0\}$ are all mutually disjoint, respectively. Whence they are finite and may be denoted by $\{J_i\}_1^m$ and $\{K_i\}_1^n$ with $J_1 < \cdots < J_m$ and $K_1 < \cdots < K_n$, respectively. Let their union be $\{I_i\}_1^N$ satisfying $I_1^- \leqslant \cdots \leqslant I_N^-$. According to Lemma 6 we assert that $\{I_i\}_1^N$ is weakly increasing and satisfies $f(I_{i+1}) = -f(I_i)$, $i = 1, ..., N-1$. ∎

Being parallel to X_f^M we given the properties of X_f^m.

LEMMA 7. *Each $I \in X_f^*$ must contain a unique interval $J \in X_f^m$.*

Proof. Put

$$J^- = \sup\{K^- : K^+ = I^+ \text{ and } K \in X_f^*\},$$
$$J^+ = \inf\{K^+ : K^- = I^- \text{ and } K \in X_f^*\},$$
$$J = (J^-, J^+).$$

The same arguments as in the proof of Lemma 5 give the one of the lemma. ▮

THEOREM 3. *X_f^m is finite. Moreover $X_f^m = \{I_i\}_1^N$ with $I_1^- \leqslant \cdots \leqslant I_N^-$ is increasing and satisfies $f(I_{i+1}) = -f(I_i)$, $i = 1, ..., N-1$.*

Proof. It is noted that if $I, J \in X_f^m$ then either $I = J$ or $I \cap J = 0$. In fact $I \neq J$ and $I \cap J \neq 0$ imply by Lemma 2 that $f(I \cap J) = f(I)$ when $f(I) = f(J)$, and that $f(I \backslash J) = f(I)$ when $f(I) = -f(J)$, contradicting $I, J \in X_f^m$. Thus we have that either $I = J$ or $I \cap J = 0$. Therefore, X_f^m is finite. Moreover X_f^m may be written as $\{I_i\}_1^N$, $I_1 < \cdots < I_N$, satisfying $f(I_{i+1}) = -f(I_i)$, $i = 1, ..., N-1$. ▮

The following theorem describes the relation between X_f^M and X_f^m where "card" denotes "the cardinality of."

THEOREM 4. card $X_f^M =$ card X_f^m, *which we denote by N_f. Furthermore, if $X_f^M = \{I_1, ..., I_{N_f}\}$ and $X_f^m = \{J_1, ..., J_{N_f}\}$ are weakly increasing, then $J_i \subset I_i$, $i = 1, ..., N_f$, and $J_i = (I_{i-1}^+, I_{i+1}^-)$, $i = 2, ..., N_f - 1$.*

Proof. By Lemmas 5 and 7 we see that card $X_f^M =$ card X_f^m and $J_i \subset I_i$, $i = 1, ..., N_f$. By Lemma 3 and Definition 1, we have that $(I_{i-1}^+, I_{i+1}^-) \in X_f^m$, $i = 2, ..., N_f - 1$. Whence $J_i = (I_{i-1}^+, I_{i+1}^-)$, $i = 2, ..., N_f - 1$. ▮

3. CHARACTERIZATION

THEOREM 5. *Let $G = \operatorname{span}\{g_1, ..., g_n\}$ be an n-dimensional subspace of $C[a, b]$, $f \in C[a, b] \backslash G$, $p \in G$, $r = f - p$ and $s(I) = \operatorname{sgn} r(I)$. Then the following statements are equivalent:*

(a) *p is a best approximation to f from G;*

(b) *There does not exist a $q \in G$ such that $s(I)\, q(I) > 0$ for all $I \in X_r$;*

(c) *The origin of n space lies in the convex hull of the set $\{s(I)\hat{I} : I \in X_r\}$, where $\hat{I} = (g_1(I), ..., g_n(I))$;*

(d) $\max_{I \in X_r} s(I)\, q(I) \geqslant 0$ *for all $q \in G$.*

Proof. It is noted that X as well as X_r are all compact. As usual, we denote by $C(X)$ the class of continuous functions on X. Then f and g_i's, as functions of I on X, are also elements of $C(X)$. Applying Theorem 1.3 of Chap. II in [3, p. 178] we directly get (a) $\Leftrightarrow$ (c). Meanwhile, since the set $\{s(I)\hat{I}: I \in X_r\}$ is a compact set of the usual n-dimensional space, according to [1, p. 19, Theorem on Linear Inequalities] we assert (b) $\Leftrightarrow$ (c). Finally, the equivalence (b) $\Leftrightarrow$ (d) is obvious. ∎

In order to establish an alternation theorem we need a further condition on $\{g_1, ..., g_n\}$, which we shall give in the following definition.

DEFINITION 3. A system of functions $\{g_1, ..., g_n\} \subset C[a, b]$ is said to be a quasi-Chebyshev system on $[a, b]$ (or a QT-system), if

$$D(I_1, ..., I_n) := \det\{g_j(I_i)\}_{i,j=1}^{n} \neq 0$$

whenever $\{I_i\}_1^n \subset X$ is increasing. An n-dimensional subspace G of $C[a, b]$ is called a QT-subspace if it has a basis which is a QT-system.

We next establish a preliminary result, which is of independent interest.

LEMMA 8. *Let $p \in C[a, b]$. Let $\{I_i\}_1^m \subset X$ be weakly increasing and $e = 1$ or -1, fixed. Suppose*

$$(-1)^i ep(I_i) \geqslant 0, \qquad i = 1, ..., m. \tag{3}$$

Then the following statements hold:

(a) *There exist m intervals $J_1, ..., J_m$, $J_1 < \cdots < J_m$, such that*

$$(-1)^i ep(J_i) \geqslant 0, \qquad i = 1, ..., m. \tag{4}$$

Furthermore, if $p(x)$ is not identically equal to zero on any nontrivial subinterval, $\{J_i\}_1^m$ may be chosen so that

$$(-1)^i ep(J_i) > 0, \qquad i = 1, ..., m; \tag{5}$$

(b) *If $m > 1$, there exist $m - 1$ intervals $K_1, ..., K_{m-1}$, $K_1 < \cdots < K_{m-1}$, such that $p(K_i) = 0$, $i = 1, ..., m - 1$.*

Proof. Assume without loss of generality that $e = 1$.

(a) Put

$$\begin{aligned}
&J_1 = I_1, && I_2' = I_2 && \text{if } I_1 \cap I_2 = 0\\
&J_1 = I_1 \backslash I_2, && I_2' = I_1 \cap I_2 && \text{if } I_1 \cap I_2 \neq 0 \text{ and } p(I_1 \cap I_2) \geqslant 0\\
&J_1 = I_1 \cap I_2, && I_2' = I_2 \backslash I_1 && \text{if } I_1 \cap I_2 \neq 0 \text{ and } p(I_1 \cap I_2) < 0.
\end{aligned}$$

It is easy to see that $p(J_1) \leqslant 0$, $p(I_2') \geqslant 0$, and $J_1 \cap I_2' = 0$. Meanwhile $\{I_2', I_3, ..., I_m\}$ is also weakly increasing and satisfies $p(I_2') \geqslant 0$ and $(-1)^i p(I_i) \geqslant 0$, $i = 3, ..., m$. By induction we can obtain $\{J_i\}_1^m$, $J_1 < \cdots < J_m$, which satisfies (4).

If $p(x)$ is not identically equal to zero on any nontrivial subinterval, then $(-1)^i p(J_i) \geqslant 0$ implies that there is a subinterval of J_i, denoted again by J_i, satisfying $(-1)^i p(x) > 0$ on J_i. Whence (5) follows.

(b) If $p(x) \equiv 0$ on some nontrivial subinterval, the conclusion is trivial. Otherwise by Part (a) there are m intervals $J_1, ..., J_m$, $J_1 < \cdots < J_m$, satisfying (5). Now choose L_i and R_i in X so that

$$L_i < R_i,\ L_i \cup R_i \subset J_i,\ (-1)^i p(L_i) > 0,\ (-1)^i p(R_i) > 0,\ i = 2, ..., m-1.$$

Since $p(I)$ is a continuous function of I, there exist $m-1$ nontrivial intervals $K_1, ..., K_{m-1}$, satisfying $p(K_i) = 0$, $i = 1, ..., m-1$ and $K_i \subset (R_i^-, L_{i+1}^+)$, $i = 1, ..., m-1$, where $R_1 = J_1$ and $L_m = J_m$. Thus $K_1 < \cdots < K_{m-1}$. ∎

We can characterize QT-systems as follows.

THEOREM 6. *Let* $G = \operatorname{span}\{g_1, ..., g_n\} \subset C[a, b]$. *Then the following statements are equivalent*:

(a) $\{g_1, ..., g_n\}$ *is a QT-system*;

(b) *For any weakly increasing intervals* $I_1, ..., I_n$,

$$D(I_1, ..., I_n) \neq 0;$$

(c) *If* $p \in G$ *satisfies* $p(I_i) = 0$, $i = 1, ..., n$, *for a weakly increasing system of intervals* $\{I_1, ..., I_n\} \subset X$, *then* $p = 0$;

(d) $\{g_1, ..., g_n\}$ *is a weak Chebyshev system on* $[a, b]$ *and every nonzero* $p \in G$ *does not vanish on any nontrivial subinterval.*

Proof. (b) $\Leftrightarrow$ (c) By means of the well known arguments.

(a) $\Rightarrow$ (c) Suppose on the contrary that $p \neq 0$ and $p(I_i) = 0$, $i = 1, ..., n$, with $\{I_i\}_1^n$ being weakly increasing. Taking x so that $\min\{I_n^-, I_{n-1}^+\} < x < I_n^+$ and denoting $J_n = (I_n^-, x)$, $J_{n+1} = (x, I_n^+)$ and $J_i = I_i$, $i = 1, ..., n-1$, we see that $J_1, ..., J_{n+1}$ are also weakly increasing and satisfy $(-1)^i ep(J_i) \geqslant 0$, $i = 1, ..., n+1$, with $e = 1$ or -1, fixed, since $p(J_n) + p(J_{n+1}) = p(I_n) = 0$. By Lemma 8 we obtain n intervals $K_1, ..., K_n$, satisfying $K_1 < \cdots < K_n$, such that $p(K_i) = 0$, $i = 1, ..., n$. Obviously $D(K_1, ..., K_n) = 0$, a contradiction.

(c) $\Rightarrow$ (d) First we easily see that every nonzero $p \in G$ does not vanish on any nontrivial subinterval. Next suppose to the contrary that $p \in G$ has n sign changes on (a, b), say, $(-1)^i p(x_i) > 0$, $i = 1, ..., n+1$, where

$x_1 < \cdots < x_{n+1}$. Thus we may choose $I_i \subset (x_i, x_{i+1})$, so that $p(I_i) = 0$, $i = 1, ..., n$, contradicting (c).

(d) $\Rightarrow$ (a) Assume that $\{g_1, ..., g_n\}$ is not a QT-system. Then there exist increasing intervals $I_1, ..., I_n$ such that $D(I_1, ..., I_n) = 0$. Hence there is a $p \in G \backslash \{0\}$ such that $p(I_i) = 0$, $i = 1, ..., n$. Since $p(x)$ is not identically equal to zero on I_i, p has at least one sign change on I_i, $i = 1, ..., n$. So we have totally at least n sign changes. This contradiction proves the implication (d) $\Rightarrow$ (a). ∎

Combining Theorem 6 and Lemma 8 the following corollary is immediate.

COROLLARY 1. *Let* $G = \operatorname{span}\{g_1, ..., g_n\} \subset C[a, b]$ *such that* $g_1, ..., g_n$ *forms a QT-system. Let* $\{I_1, ..., I_{n+1}\} \subset X$ *be weakly increasing and* $e = 1$ *or* -1, *fixed. If* $p \in G$ *satisfies* $(-1)^i\, e p(I_i) \geqslant 0$, $i = 1, ..., n+1$, *then* $p = 0$.

From Theorem 6 we obtain directly

COROLLARY 2. *A Chebyshev system must be a QT-system.*

LEMMA 9. *Let G be an n-dimensional QT-subspace of* $C[a, b]$. *Let a system of extended intervals* $\{I_i\}_1^m := \{I'_j\} \cup \{x_k\}$ *be increasing, where* $\{I'_j\} \subset X$ *and* $\{x_k\} \subset (a, b)$. *Suppose* $m < n$. *Then there exists a nonzero function* $p \in G$ *such that*

(a) $p(I_1) = 0$, $i = 1, ..., m$;

(b) *p changes sign on each* I_i, $i = 1, ..., m$ (*if* $I_i = x_k$, *this means that p changes sign at* x_k);

(c) *p has exactly m sign changes on* $[a, b]$.

Proof. Put for $t > 0$ sufficiently small

$$J_i = \begin{cases} (b-(n-i)t,\, b-(n-i-1)t), & i = m+1, ..., n-1 \\ \qquad \text{if } \; m < n-1 & \\ (x_i - t,\, x_i + t) \qquad \text{if } \; I_i \in \{x_k\} & \\ I_i \backslash \left\{ \left(\bigcup_l \,[b-(n-l)t,\, b-(n-l-1)t] \right) \cup \left(\bigcup_k \,[x_k - t,\, x_k + t] \right) \right\} & \\ \qquad \text{if } \; I_i \in \{I'_j\}. & \end{cases}$$

We see that $\{J_i\}$ is also increasing if $t > 0$ is sufficiently small. Since G is a QT-subspace, there exists a nonzero function $p_t \in G$ such that $p_t(J_i) = 0$, $i = 1, ..., n-1$, p changes sign on each J_i, $i = 1, ..., n-1$ and has no sign change in each interval (J_i^+, J_{i+1}^-), $i=0, ..., n-1$, where $J_0^+=a$ and $J_n^-=b$. Furthermore we assume that $\|p_t\|=1$. Letting $t \downarrow 0$, we select a limit function $p \in G$ satisfying

(1) $\|p\|=1$;

(2) $p(I_i)=0$, $i=1, ..., m$;

(3) p does not change sign in each interval (I_i^+, I_{i+1}^-), $i=0, ..., m$, where $I_0^+=a$ and $I_{m+1}^-=b$. It is easy to see that p changes sign on each I_i, $i=1, ..., m$ and has exactly m sign changes. This completes the proof. ∎

The main result in the present section is as follows.

THEOREM 7. *Let* $G=\operatorname{span}\{g_1, ..., g_n\} \subset C[a, b]$ *be an n-dimensional QT-subspace. Let*

$$f \in C[a, b] \backslash G, \qquad p \in G, \qquad r=f-p, \qquad s(I)=\operatorname{sgn} r(I).$$

Then the following statements are equivalent:

(a) *p is a best approximation to f from G*;

(b) *There does not exist a* $q \in G$ *such that* $s(I)\, q(I)>0$ *for all* $I \in X_r$;

(c) *The origin of n space lies in the convex hull of the set* $\{s(I)\hat{I}: I \in X_r\}$, *where* $\hat{I}=(g_1(I), ..., g_n(I))$;

(d) $\max_{I \in X_r} s(I)\, q(I) \geqslant 0$ *for all* $q \in G$;

(e) $\max_{I \in X_r} s(I)\, q(I)>0$ *for all* $q \in G \backslash \{0\}$;

(f) $N_r \geqslant n+1$.

Moreover, the conclusions remain true if we replace X_r *by any one of* X_r^*, X_r^M, *and* X_r^m.

Proof. Theorem 5 already contains the equivalences (a) ⇔ (b) ⇔ (c) ⇔ (d). We now show the other equivalences. Denote $N=N_r$ and $X_r^m=\{I_1, ..., I_N\}$ with $I_1<\cdots<I_N$. Assume without loss of generality that $s(I_1)>0$.

(b) ⇒ (f) Suppose to the contrary that $N \leqslant n$. Put

$$K_i=\begin{cases}(I_i^-, I_{i+1}^+) & \text{if } i=\text{odd} \\ (I_i^+, I_{i+1}^-) & \text{if } i=\text{even and } I_i^+<I_{i+1}^- \\ I_i^+ & \text{if } i=\text{even and } I_i^+=I_{i+1}^-.\end{cases} \quad (i=1, ..., N-1).$$

Obviously the system of extended intervals $\{K_1, ..., K_{N-1}\}$ is increasing. By Lemma 9 there is a nonzero $q \in G$ such that (1) $q(K_i)=0$, $i=1, ..., N-1$; (2) q changes sign on each interval K_i, $i=1, ..., N-1$; (3) q has exactly

$N-1$ sign changes on $[a, b]$. We assume that $q(I_1) \geqslant 0$ (taking $-q$ instead of q if necessary). Denote $K_0 = [a, K_1^-)$ and $K_N = (K_{N-1}^+, b]$.

Assertion. If K_i is nontrivial for $0 \leqslant i \leqslant N$ then

$$(-1)^{i+1} q((K_i^-, x)) > 0, \qquad x \in K_i,\ i > 0$$

and

$$(-1)^{i+1} q((x, K_i^+)) < 0, \qquad x \in K_i,\ i < N.$$

There are three cases to be discussed.

Case 1. $0 < i < N$.

In this case it follows from $q(K_i) = 0$ that

$$q((K_i^-, x)) = -q((x, K_i^+)).$$

Since $q(I_1) \geqslant 0$ and q has exactly one sign change on K_i,

$$(-1)^{i+1} q((K_i^-, x)) > 0$$

and

$$(-1)^{i+1} q((x, K_i^+)) < 0.$$

Especially, for $i = 1$ and $i = N-1$ we obtain

$$q((K_1^-, x)) > 0 \tag{6}$$

and

$$(-1)^N q((x, K_{N-1}^+)) < 0. \tag{7}$$

Case 2. $i = 0$.

Since q has no sign change on K_0, by (6) we obtain $q((x, K_0^+)) > 0$. This proves the assertion when $i = 0$.

Case 3. $i = N$.

Since q has no sign change on K_N, if $K_{N-1} \notin \{x_k\}$ we obtain by (7) that $(-1)^N q((K_N^-, x)) < 0$ or $(-1)^{N+1} q((K_N^-, x)) > 0$, which is the assertion when $i = N$. Clearly this assertion is also valid for $K_{N-1} \in \{x_k\}$.

Now let $I \in X_r$ be arbitrary. Then the interval I must contain an odd number of I_i's, say, $I \supset (I_j \cup \cdots \cup I_{j+2k})$, where $j \geqslant 1$, $j + 2k \leqslant N$, $k \geqslant 0$. Thus $I \supset (K_j \cup \cdots \cup K_{j+2k-1})$. Letting $L = (I^-, K_{j-1}^+)$ and $R = (K_{j+2k}^-, I^+)$, we have that $q(I) = q(L) + q(K_j \cup \cdots \cup K_{j+2k-1}) +$

$q(R)=q(L)+q(R)$. If $L\neq 0$ and $I^-\in K_{j-1}$ then $(-1)^j q(L)<0$, i.e., $(-1)^{j+1} q(L)>0$; otherwise $q(L)=0$. Also, if $R\neq 0$ and $I^+\in K_{j+2k}$ then $(-1)^{j+2k+1} q(R)>0$, i.e., $(-1)^{j+1} q(R)>0$; otherwise $q(R)=0$. Thus $(-1)^{j+1}$ $q(I)>0$ since $q(L)=0$ and $q(R)=0$ may not occur simultaneously. According to the assumption that $s(I_1)>0$ we conclude that $s(I)=s(I_j)=(-1)^{j+1} s(I_1)=(-1)^{j+1}$ and whence $s(I)\, q(I)>0$, contradicting (b).

(f) ⇒ (e) Assume (c) does not hold and let $q\in G\backslash\{0\}$ satisfy $\max_{I\in X_r}$ $s(I)\, q(I)\leqslant 0$. Whence $\max_{I\in X_r^m} s(I)\, q(I)\leqslant 0$ or $s(I_i)\, q(I_i)\leqslant 0$, $i=1, ..., N$. Since $s(I_i)=(-1)^{i+1} s(I_1)$,

$$(-1)^i s(I_1)\, q(I_i)\leqslant 0,\ i=1, ..., N.$$

By Corollary 1, $q=0$, a contradiction.

(e) ⇒ (d) Trivial.

In the proof of (f) ⇒ (e) we have actually shown that (f) implies $\max_{I\in X_r^m} s(I)\ q(I) > 0$ for all $q \in G\backslash\{0\}$. Similarly, (f) implies $\max_{I\in X_r^M} s(I)\, q(I)>0$ for all $q\in G\backslash\{0\}$ and implies $\max_{I\in X_r^*} s(I)\, q(I)>0$ for all $q\in G\backslash\{0\}$. On the other hand, the implications (e) ⇒ (d) ⇒ (c) ⇒ (b) ⇒ (a) ⇒ (f) remain valid if we replace X_r by any one of X_r^m, X_r^M, and X_r^*. ∎

THEOREM 8. *Let* $G=\text{span}\{g_1, ..., g_n\}\subset C[a, b]$ *be an n-dimensional QT-subspace and* $f\in C[a, b]\backslash G$. *Let* $p\in G$ *satisfy*

$$(-1)^i e(f(I_i)-p(I_i))\geqslant 0, \qquad i=1, ..., n+1, \tag{8}$$

where $\{I_i\}\subset X$, $I_1<\cdots<I_{n+1}$, *and* $e=1$ *or* -1, *fixed. Then*

$$\inf_{q\in G} \|f-q\|\geqslant \min_{1\leqslant i\leqslant n+1} |f(I_i)-p(I_i)|.$$

Equality can occur if and only if p is a best approximation to f and $\{I_i\}\subset X_{f-p}$.

Proof. Letting $p^*\in G$ be a best approximation to f,

$$\|f-p^*\|\leqslant \min_{1\leqslant i\leqslant n+1} |f(I_i)-p(I_i)|$$

implies that

$$(-1)^i e(p^*(I_i)-p(I_i))\geqslant 0, \qquad i=1, ..., n+1.$$

By Corollary 1 we must have $p=p^*$ and $\{I_i\}\subset X_{f-p}$. Conversely, if p is a best approximation to f and $\{I_i\}\subset X_{f-p}$ then equality occurs. ∎

4. Uniqueness

Theorem 9. *Let p be a best approximation from G to $f \in C[a, b]$. If G is a QT-subspace of $C[a, b]$, then p is unique.*

Proof. If $f \in G$ then $p = f$ is unique. Now suppose $f \notin G$. Let $p^* \in G$ be another best approximation. Then for $X^m_{f-p} = \{I_1, ..., I_{N_{f-p}}\}$, $I_1 < \cdots < I_{N_{f-p}}$, we have (8) with $e = -\operatorname{sgn}(f(I_1) - p(I_1))$ and $\|f - p^*\| = \|f - p\| = \min\{|f(I_i) - p(I_i)| : 1 \leqslant i \leqslant N_{f-p}\}$. From Theorem 8 it follows that $p = p^*$. ∎

By the same arguments as in the proof of [1, p. 80, Strong Unicity Theorem] we obtain the following.

Theorem 10. *Let p be a best approximation from G to $f \in C[a, b]$. If G is an n-dimensional QT-subspace of $C[a, b]$, then there exists a constant $\gamma > 0$ depending on f such that for any $q \in G$*

$$\|f - q\| \geqslant \|f - p\| + \gamma \|p - q\|.$$

Let G be an n-dimensional QT-subspace of $C[a, b]$. Then to each $f \in C[a, b]$ let $\tau f \in G$ be the (unique) best approximation to f. An analysis similar to the proof of the theorem in [1, p. 82] *gives*

Theorem 11. *Let G be an n-dimensional QT-subspace of $C[a, b]$. Then to each $f_0 \in C[a, b]$ there corresponds a number $\lambda > 0$ such that for all $f \in C[a, b]$*

$$\|\tau f - \tau f_0\| \leqslant \lambda \|f - f_0\|.$$

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